Reordering Braid Words

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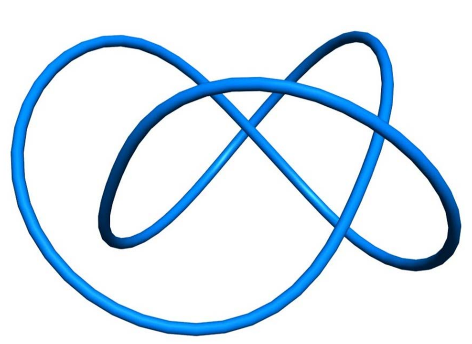
Abstract

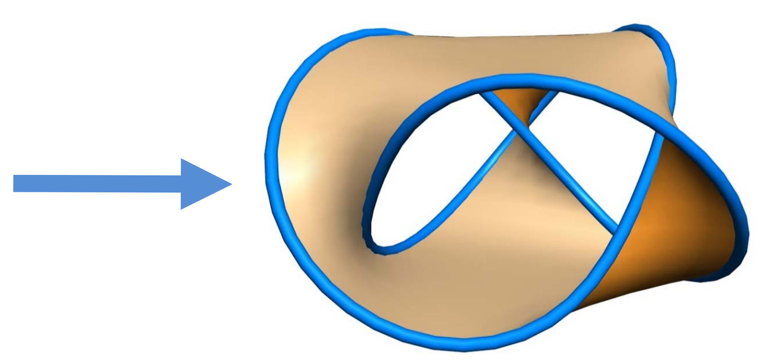
We develop and implement functions that receive braid words and reorder them in an attempt to optimize the stability of their associated Seifert Surfaces. We can attempt to gain intuition about a knot or Seifert Surface by using a 3D-printed model which is why increasing its stability is important.

What is a Seifert Surface?

A Seifert Surface is an oriented surface in 3-space whose boundary is a given knot. They exist for every knot but are not unique.



Trefoil Knot



Seifert Surface with Trefoil Knot Boundary

What is a braid?

A braid is composed of $2n$ fixed points on a rectangle in the plane, n points above which connect to n points below. The curves connecting the points can only flow down. A braid can be represented as a word formed by braid letters. Braid letters are represented by sigma and a subscript (e.g. σ_i) where i is an integer between 1 and $(n-1)$ inclusive. The letter σ_i means that the line from point i makes a crossing with the line from point $(i+1)$. Braid words are stacked so that leftmost letter is on the top of the braid. E.g. Figure 1 would be $\sigma_1\sigma_3\sigma_1\sigma_4^{-1}\sigma_2$ etc ...

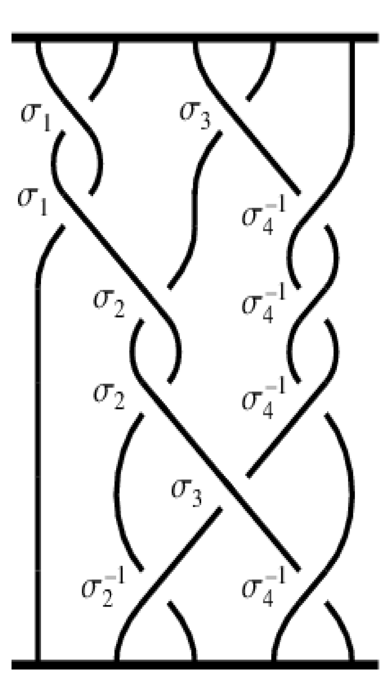


Figure 1

Braids are multiplied by stacking. When we can reorder, there are some rules we must use. These are the two we will look at:

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$.
2. $\sigma_i\sigma_{(i+1)}\sigma_i = \sigma_{(i+1)}\sigma_i\sigma_{(i+1)}$

To simplify the braid word, we write a series of numbers which correspond to the respective subscript of the sigma.

E.g. Figure 1 would be (1, 3, 1, -4, 2,...)

Design and Implementation Discussion

Stability of a structure is evaluated in different ways for each evaluation method. In the first method, it is measured by how evenly crossings in each level spread because a symmetric structure provides balance. The second method evaluates stability by analyzing the torque. The third method rates stability by how evenly subgroups spread between layers which relates to symmetry of subgroups of the structure. The evaluations of stability are subjective and there is no perfect answer to what makes a structure stable. Each of these evaluations has its weaknesses that must be counteracted by other methods.

Evaluation I

This evaluation measures the symmetry of bands in each level that keeps a physical object's balance. It first calculates the optimal distance between any two crossings on the same level. Then, it measures how close the radial distances between crossings are to the optimal distance and produces a rating. The lower levels are prioritized in this process, because a stable foundation is crucial for certain 3D-printing methods.

$$U = \frac{\sum_{a=0}^{n}|d_a - 1|}{n^2} \times 100\%$$

$$W = \frac{2^{(i_{max}-1)}}{2^{i_{max}} - 1} \cdot \left(\sum_{i=1}^{n} \frac{U}{2^{(i-1)}}\right)$$

where, $d_o =$ *optimal distance* $= \frac{2\pi}{n}$;
$i =$ *the braid strand/level*;
$n =$ *the total number of crossings*;
$d_a =$ *actual distance between two crossings*;
$i_{max} =$ *number of strands in the braid minus* 1.

Evaluation II

This evaluation determines the most stable structure based on the torque on each crossing. This allows us to discuss stability in terms of structural characteristics of a physical object. It assumes that the mass of each disc and each crossing is 1 and that each disc has a radius of 1. It then adds up the torque of each crossing and uses the total torque to rate the structure's stability. At each crossing, the torque the torque is calculated via:

Let $CM = (x, y)$

$$(x,y) = \left(1.5 \cdot \sum_{p=1}^{n}(\cos\alpha_p),\ 1.5 \cdot \sum_{p=1}^{n}(\sin\alpha_p)\right)$$

$$\vec{c} = \frac{\sqrt{x^2+y^2}}{2} \times (n - i - 1 + A_b)$$

$$\vec{d} = \sqrt{\left(\sum_{p=1}^{n}(\cos\beta_p - x)\right)^2 + \left(\sum_{p=1}^{n}(\sin\beta_p - y)\right)^2} \times (n - i + B_b)$$

$$|\vec{\tau}| = \sqrt{c^2 + d^2 - 2|\vec{c}|\left|\vec{d}\right| \cdot \cos\left|\arctan\left(\frac{y}{x}\right) - \arctan\left(\frac{\sum_{i=1}^{n}\sin\beta_i}{\sum_{i=1}^{n}\cos\beta_i}\right)\right|}$$

where,
$CM =$ *center of mass of a disc*
$p =$ *position of a crossing in the braid word*
$\alpha_p, \beta_p =$ *angle between crossing band and the standard band above/below*
$A_b, B_b =$ number of bonds below level i / $(i-1)$
$\vec{c}, \vec{d} =$ *the net torque of forces downward / upward*
$\vec{\tau} =$ the net torque

NOTE:
CM of a bond is assumed as 1.5 far from the center of disk.

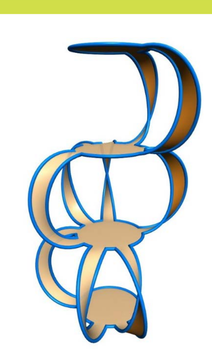
Evaluation III

This evaluation partitions the braid word into groups in which each group has no more repeated indexes. For example, (1,2,2,3,1,2) becomes (1,2)(2,3,1)(2). The formula below calculates the optimal size of each group and then compares the actual size to the optimal to rate its stability.

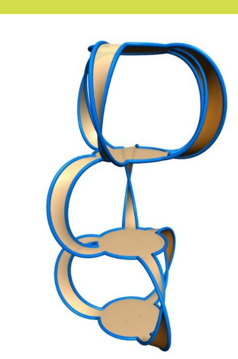
$$Q = \sum_{s=1}^{n}\left|\frac{N_b}{i_{max}} - N_s\right|$$

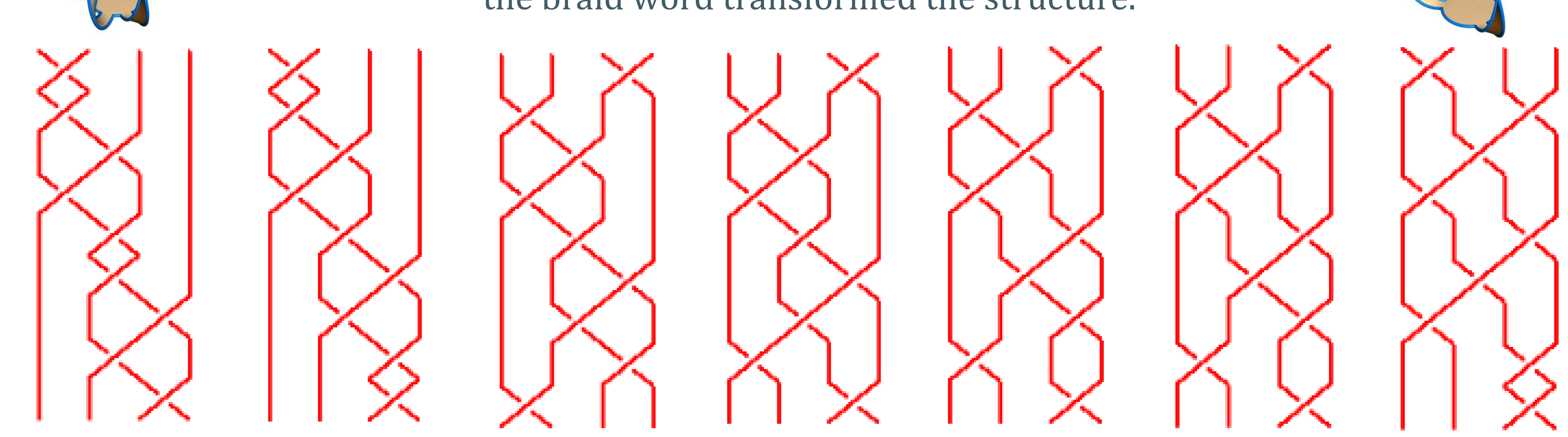
where, $s =$ *set*;
$n_s =$ *number of braids in xth set*;
$N_b =$ *total number of bonds*.

Example



This is one result of the shelling algorithm with Evaluation II. The red numbers indicate changes in the braid word, but the braid remains the same. The Seifert Surfaces above show how changing the braid word transformed the structure.

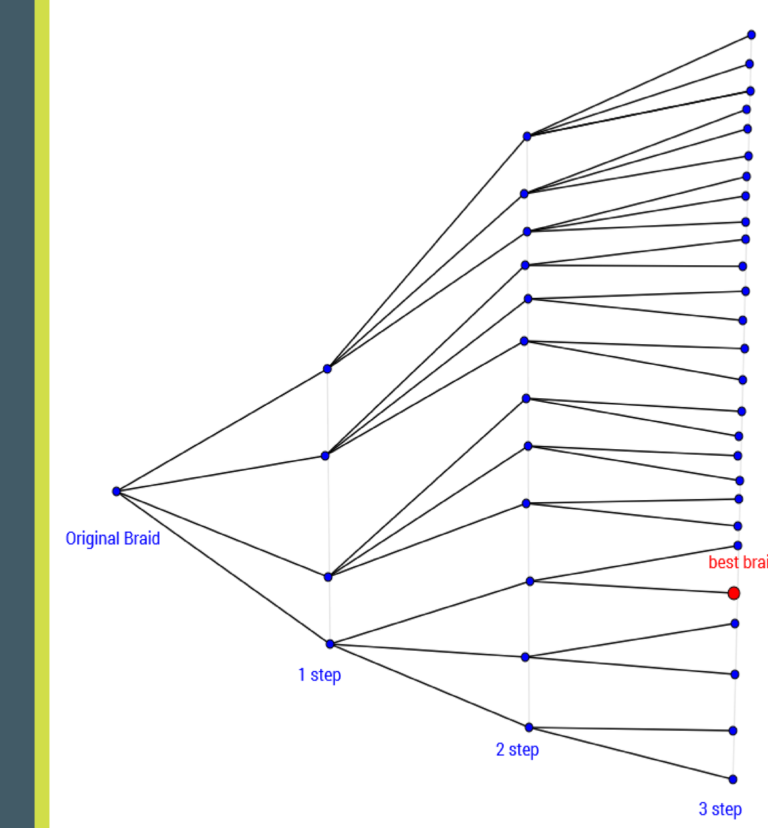




[1, 1, 2, 1, 2, 2, 3, 2, 3] [1, 1, 2, 1, 2, 3, 2, 3, 3] [3, 1, 2, 1, 2, 3, 2, 3, 1] [3, 1, 2, 1, 2, 3, 2, 1, 3] [3, 1, 2, 1, 3, 2, 3, 1, 3] [3, 1, 2, 1, 3, 2, 1, 3, 3] [1, 3, 2, 1, 3, 2, 1, 3, 3]

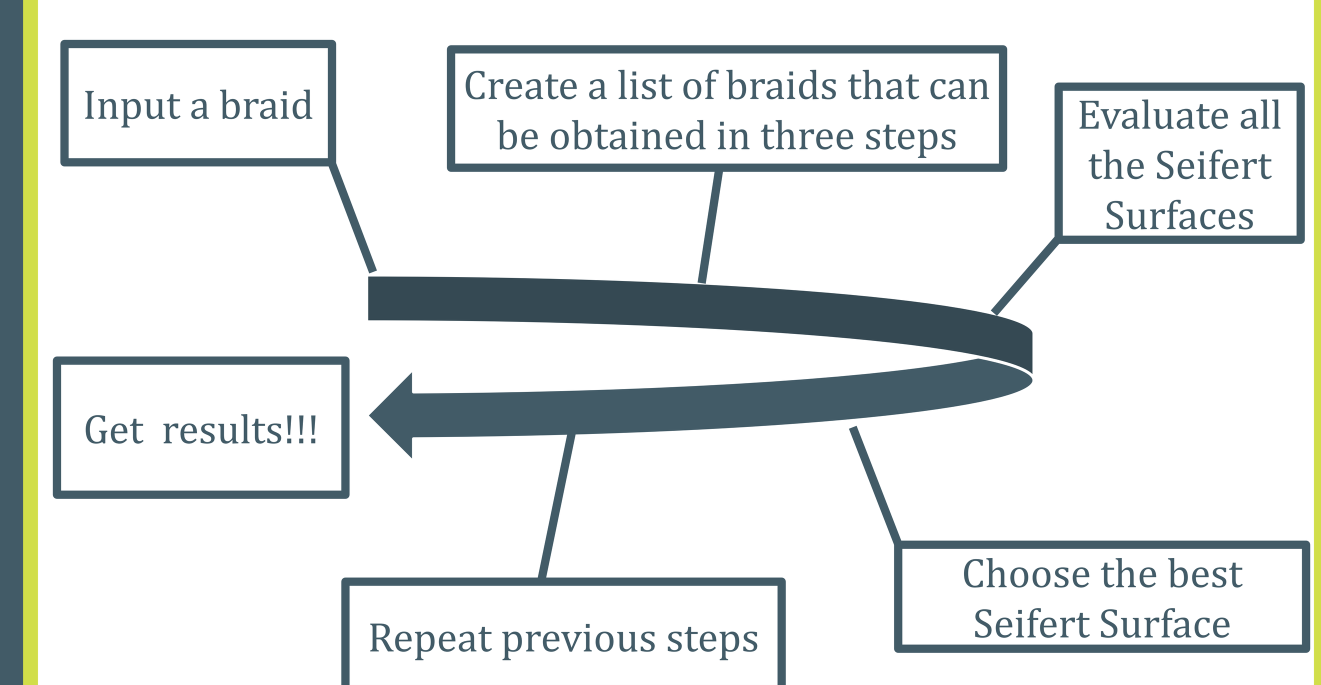
Shell Algorithm

The program produces a list of braids that can be obtained in three steps with the rules of reordering. With a given evaluation function, it evaluates all reorderings within three changes of the original braid. After comparing all of the results, it selects the best reordering as determined by the evaluation and the process is repeated until there is no better reordering.



Our program searches for the best braid from all the braids that can be created in three steps with the two basic reordering rules. Using this method, we can only find the locally "ideal" braid. To find the global one, we would need to compare all the possible braid reorderings.

Pseudocode



Further Research

- We have already derived three possible methods. How can we combine them into system that avoids each method's weakness?
- What methods can we use to evaluate the Seifert Surface if there are more factors involved, like smoothness, length, and thickness of crossings and discs?
- How can the shell algorithm be improved to find the global maximum without searching every possible reordering?
- Can more rules be used to reorder the braid word and thus find a new Seifert Surface that was not previously considered?

Acknowledgments

We would like to thank Dr. Kyle Istvan, Mrs. Crum, Spencer Roby, teaching assistants Jenny & Jeffery, and everyone else who made the Math Circle possible.